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AN ERROR ESTIMATION OF AN ANALYTIC AND NUMERICAL SOLUTION OF A HEAT EQUATION USING FINITE DIFFERENCE SCHEME

Anupam Hayath Chowdhury*

Abstract

We study finite difference scheme for second order partial differential heat equation, which is a parabolic partial differential equation .We report the analytical solution of the partial differential equation as a heat problem by the method of characteristic and also numerical solution with the help of finite difference scheme, where the solution reads an explicit form and depends on the initial value. Therefore one needs to use numerical methods for solving the second order partial differential equation as an initial boundary value problem. For this we present an explicit finite difference scheme and implement the numerical scheme by computer programming for various cases of heat equation.

Introduction

The subject of Differential Equations is a well established part of mathematics and its systematic development goes back to the early days of the development of Calculus. Many recent advances in mathematics, paralleled by a renewed and flourishing interaction between mathematics, the sciences, and engineering, have again shown that many phenomena in the applied sciences, modeled by differential equations will yield some mathematical explanation of these phenomena (at least in some approximate sense).

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines. As differential equations are equations which involve functions and their derivatives as unknowns, we shall adopt throughout the view that differential equations are equations in spaces of functions.

* Lecturer in Mathematics, Asian University of Bangladesh, Uttara, Dhaka.

Objectives of the Study

(a) To find out the analytical error

(b) To find out the numerical error

(c) To give the solution for both error

(d) T view the final result of an error

Limitations of the Study

(a) Lack of adequate knowledge about mathematical formula it may not give perfect result

(b) Lack of time it may cause minor imperfection

(c) Researcher could not give enough financial effort here that my reflect on the research

Problem

Here we study finite difference schemes for first order Cauchy Linear PDE which is treated as an instant BVP as an initial BVP

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

With initial and boundary condition

$$u(0,x) = u_0(x)$$
$$u(t,a) = u_a(t)$$
$$u(t,b) = u_b(t)$$

In this case, we discuss four finite difference schemes such as:

(a) Explicit upwind difference scheme

(b) Explicit downwind difference scheme

First we descretize the PDE by finite difference formula for which leads to formulate the finite diffrence schemes. We also discuss the stability condition of the finite difference schemes and compare their stability condition with respect to the advantages and disadvantages. Finally we developed a code and implement the explicit upwind difference scheme for first order Cauchy Linear PDE using math lab programming language.

Finite difference Methods:

We will develop approximation to solutions of general nonlinear scalar conservation laws on a bounded domain.

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \text{ for } a<x<b, 0<t \qquad \ldots\ \ldots\ \ldots..4(a)$$

Initially, we will assume that the characteristics speeds satisfy $\lambda = f'(a) > 0$ for all states that occur in the solution of (3.2-1a). Thus, we also provide initial data

$$u(x,0)=u_0(x) \qquad \ldots\ \ldots\ \ldots.4(b)$$

and data at the left hand boundary

$$u(a,t)=u_a(t) \qquad \ldots\ \ldots\ldots.\ 4(c)$$

In order to approximate the solution to (3.2-1), we will discretize space

$$a=x_{-\frac{1}{2}}<x_{\frac{1}{2}}<\ldots<x_{1-\frac{3}{2}}<x_{1-\frac{1}{2}}=b$$

and time

$$0=t^0<t^1<\ldots<t^{N-1}<t^N=1.$$

We will define the computational grid cells to be the intervals $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, with cells widths

$$\Delta x_i \equiv x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \ .$$

We will also define the time step to be

$$\Delta t^{n+\frac{1}{2}} \equiv t^{n+1}-t^n \ .$$

(See figure 4.1 for an illustration of spatial and temporal discretization) We will integrate the differential equation 4(a) over the space time rectangle $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})\times(t^n, t^{n+1})$ to get

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^{n+1})dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx + \int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}},t))dt - \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))dt \ .$$

This equation involves no approximations

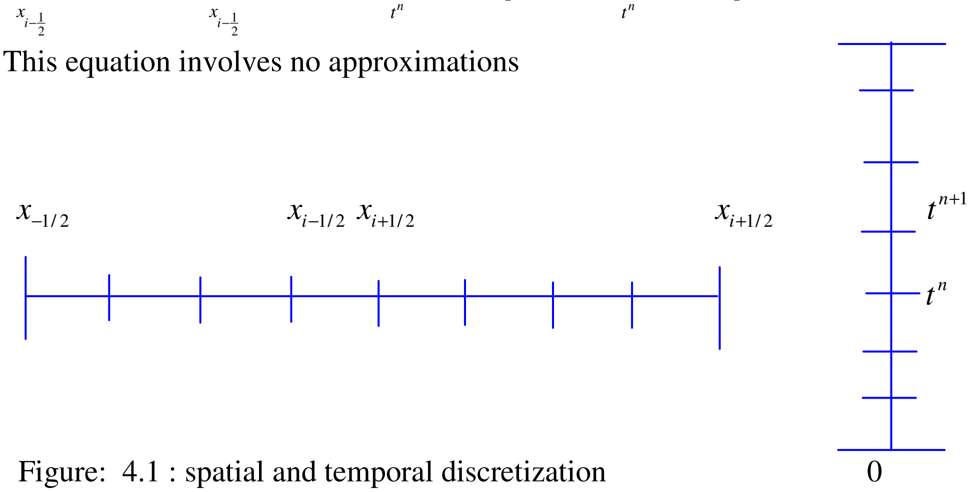


Figure: 4.1 : spatial and temporal discretization

We will discretize the conservation law by approximating the conserved densities

$$u_i^n \approx \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^n) dx$$

and the fluxes

$$\int_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta t^{n+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}}, t)) dt$$

We will require our discretization to satisfy the conservative difference

$$u_i^{n+1} = u_i^n - \frac{\Delta t^{n+\frac{1}{2}}}{\Delta x_i} [\int_{i+\frac{1}{2}}^{n+\frac{1}{2}} - \int_{i-\frac{1}{2}}^{n+\frac{1}{2}}].$$

Initially, we define

$$u_i^0 = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x) dx .$$

Further, at left-hand boundary $x_{-\frac{1}{2}} = a$ we define

$$f_{-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta t^{n+\frac{1}{2}}} \int_{t^n}^{t^{n+1}} f(u_a(t)) dt$$

Conservative differences conserve the total discrete conserved quantity:

$$\sum_{i=0}^{t-1} u_i^{n+1} \Delta x_i = \sum_{i=0}^{t-1} u_i^n \Delta x_i - \Delta t^{n+\frac{1}{2}} [f_{1-\frac{1}{2}}^{n+\frac{1}{2}} - f_{-\frac{1}{2}}^{n+\frac{1}{2}}] .$$

Distinct choices of conservatives schemes differ primarily in the method of approximating the discrete fluxes.

The physical domain of dependence of the cell average u_i^{n+1} can be determined by tracing back characteristics from the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ at time t^{n+1} to some spatial interval at time t^n .(We know how to trace back along characteristics for linear advection: we will learn how to handle nonlinear conservation laws in section 3.3). In order for the numerical scheme to converge to the analytical solution of the conservation law, it is necessary that the numerical domain of dependence of u_i^{n+1} contain the physical domain of dependence. We will examine this condition for several examples of methods in the following subsections.

Explicit Upwind Difference scheme:

The linear scalar conservation law or linear advection equation is

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0 \qquad ....4(d)$$

With initial condition

$$u(x,0)=u_0(x)$$

And left-hand boundary condition:

$$u(x,t)=u_a(t)$$

We discrete the space

$$a=x_{-\frac{1}{2}}<x_{\frac{1}{2}}<......<x_{l-\frac{3}{2}}<x_{l-\frac{1}{2}}=b$$

And time

$$0=t^0<t^1<t^2<........<t^{N-1}<t^N=T$$

Let the spatial and temporal grid sizes be

$$\Delta x_i \equiv x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}} \;;\; i=0,1,........,l-1$$

$\Delta t^{n+\frac{1}{2}} \equiv t^{n+1}-t^n \;;\; n=0,1,........,\mathrm{N}-1$ The possible finite difference approximation for $\frac{\partial u}{\partial x}$ are:

Forward difference:

From Taylor's series

$$u(x+h)=u(x)+hu'(x)+\frac{h^2}{2!}u''(x)+\cdots\cdots$$

$$\Rightarrow u'(x)=\frac{u(x+h)-u(x)}{h}-o(h)$$

$$\Rightarrow u'(x)\approx\frac{u(x+h)-u(x)}{h}$$

Similarly, Backward difference:

$$u'(x)\approx\frac{u(x)-u(x-h)}{h}$$

The discretization of $\frac{\partial u}{\partial t}(x_i^n)$ is obtained by first order forward difference in time

$$\frac{\partial u}{\partial t}(x_i^n) \approx \frac{u_i^{n+1}-u_i^n}{\Delta t^{n+1/2}}$$

The discretization of $\frac{\partial u}{\partial x}(x_i^n)$ is obtained by first order backward difference in space

$$\frac{\partial u}{\partial x}(x_i^n) \approx \frac{u_i^n - u_{i-1}^n}{\Delta x_i}$$

Substituting these values in equation (1) then we have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t^{n+1/2}} + \mathrm{c}\frac{u_i^n - u_{i-1}^n}{\Delta x_i} = 0 \quad = 0$$

$$u_i^{n+1} = u_i^n - \frac{\mathrm{c}\Delta t^{n+1/2}}{\Delta x_i}(u_i^n - u_{i-1}^n)$$

Which is known as explicit upwind difference scheme of 4(d).

Stencil:

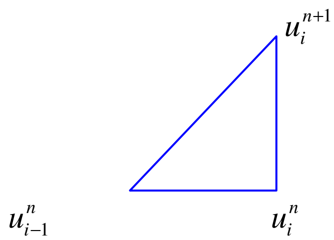


Figure 4.2: Stencil of explicit upwind difference scheme.

Stability Condition:

The simplest numerical approximation to the linear advection equation of explicit upwind difference scheme is

$$u_i^{n+1} = u_i^n - \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i}\left[u_i^n - u_{i-1}^n\right]$$

This is a conservative difference scheme in which $f_{i+\frac{1}{2}}^{n+\frac{1}{2}} = cu_i^n$. In this case the domain of dependence of u_i^{n+1} is the interval $(x_{i-\frac{3}{2}}, x_{i+\frac{1}{2}})$ corresponding the cell average u_i^n and u_{i-1}^n. The physical domain of dependence is the interval $(x_{i-\frac{1}{2}} - c\Delta t^{n+\frac{1}{2}}, x_{i+\frac{1}{2}} - c\Delta t^{n+\frac{1}{2}})$.

The numerical domain of dependence contains the physical domain of dependence iff $x_{i-\frac{3}{2}} \le x_{i-\frac{1}{2}} - c\Delta t^{n+\frac{1}{2}}$ and $x_{i+\frac{1}{2}} - c\Delta t^{n+\frac{1}{2}} \le x_{i+\frac{1}{2}}$.

$$\Leftrightarrow c\Delta t^{n+\frac{1}{2}} \leq \min(\Delta x_i)\forall i = 0,1,2,........,l-1$$

$$\Leftrightarrow \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i} \leq 1 \forall i = 0,1,2,..................,l-1$$

Which is known as the Courant-Friedrichs-Levy condition (i.e. CFL).

This suggest to define the dimensionless courant number $\gamma_i^{n+\frac{1}{2}} := \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i}$.

Rewrite the E.U.D.S in terms of courant number

$$u_i^{n+1} = (1-\gamma_i^{n+\frac{1}{2}})u_i^n + u_{i-1}^n \gamma_i^{n+\frac{1}{2}}, \forall i = 0,1,2,.......,l-1 ....4(e)$$

Now if we choose $\gamma_i^{n+\frac{1}{2}} \leq 1$ then 4(e) implies that the new solution is an average of the previous values, the extreme value of the solution at the new time lie between the extreme values of the previous solution. This means that the E.U.D.S is stable when the CFL condition is satisfied i.e. $\Leftrightarrow N_d \supseteq P_d$.

Explicit Downwind Difference scheme:

The linear scalar conservation law or linear advection equation is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0;\ x \in (a,b), t \in (0,T) \qquad ....4(f)$$

With initial condition

$$u(x,0) = u_0(x)$$

And right-hand boundary condition:

$$u(a,t) = u_a(t)$$

We discretize the space

$$a = x_{-\frac{1}{2}} < x_{\frac{1}{2}} < ...... < x_{l-\frac{3}{2}} < x_{l-\frac{1}{2}} = b$$

And time

$$0 = t^0 < t^1 < ..... < t^{N-1} < t^N = T$$

Let the spatial and temporal grid sizes be

$$\Delta x_i \equiv x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\ ;\ i = 0,1,........,l-1$$

$$\Delta t^{n+\frac{1}{2}} \equiv t^{n+1} - t^n\ ;\ n = 0,1,........,\mathrm{N}-1$$

The possible finite difference approximation for $\frac{\partial u}{\partial x}$ are:

Forward difference:
From Taylor's series

$$u(x+h)=u(x)+hu'(x)+\frac{h^2}{2!}u''(x)+\cdots\cdots$$

$$\Rightarrow u'(x)=\frac{u(x+h)-u(x)}{h}-o(h)$$

$$\Rightarrow u'(x)\approx\frac{u(x+h)-u(x)}{h}$$

The discretization of $\frac{\partial u}{\partial t}(x_i^n)$ is obtained by first order forward difference in time

$$\frac{\partial u}{\partial t}(x_i^n) \approx \frac{u_i^{n+1}-u_i^n}{\Delta t^{n+1/2}}$$

The discretization of $\frac{\partial u}{\partial x}(x_i^n)$ is obtained by first order forward difference in space

$$\frac{\partial u}{\partial x}(x_i^n) \approx \frac{u_{i+1}^n-u_i^n}{\Delta x_i}$$

Substituting these values in equation (1) then we have

$$\frac{u_i^{n+1}-u_i^n}{\Delta t^{n+1/2}}+c\frac{u_{i+1}^n-u_i^n}{\Delta x_i}=0 \quad =0$$

$$u_i^{n+1}=u_i^n-\frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i}(u_{i+1}^n-u_i^n) \qquad .... \quad .... \quad ....4(g)$$

Which is known as explicit downwind difference scheme of 4(f).

Stencil:

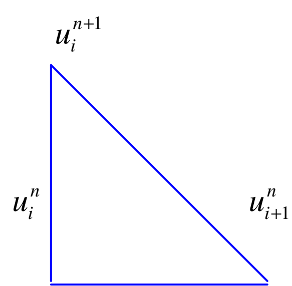


Figure 4.3 : Stencil of explicit downwind difference scheme.

Stability Condition:

The explicit downwind difference scheme for linear advection equation is

$$u_i^{n+1} = u_i^n - \frac{c\Delta t^{n+\frac{1}{2}}}{\Delta x_i}\left[u_{i+1}^n - u_i^n\right], i = 0,1,2,.....,l-2$$

This is a conservative difference scheme in which $f_{i+\frac{1}{2}}^{n+\frac{1}{2}} = cu_{i+1}^n$. In this case the domain of dependence of u_i^{n+1} is the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ corresponding to the cell average u_i^n and u_{i+1}^n. The physical domain of dependence is the interval $(x_{i-\frac{1}{2}} - c\Delta t^{n+\frac{1}{2}}, x_{i+\frac{1}{2}} - c\Delta t^{n+\frac{1}{2}})$, corresponding to tracing the endpoints of the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ backward along characteristics from t^{n+1} to t^n.

The numerical domain of dependence never contains the physical domain of dependence, no matter what the step size of time step $\Delta t^{n+\frac{1}{2}}$.

The explicit downwind scheme can be rewrite in terms of courant number is

$$u_i^{n+1} = (1+\gamma_i^{n+\frac{1}{2}})u_i^n - u_{i+1}^n \gamma_i^{n+\frac{1}{2}}$$

Of course, we can ignore the scheme at the left hand boundary and take u_0^{n+1} to determine by the data. This gives us a right-triangular system of equation for the new solution. Because the new solution involves amplification of some information from the previous time the upwind scheme allows for instability to develop. It is also important note that the upwind scheme does not use the boundary data $u_a(t)$.

Comparison of Explicit Upwind and Downwind Difference Scheme:

Upwind is scheme is applicable for the left hand boundary, it doesn't work at the right hand boundary. It is easy to incorporate and it is stable when CFL condition is satisfied.

$$\text{i.e } \gamma_i^{n+\frac{1}{2}} = c\frac{\Delta t^{n+\frac{1}{2}}}{\Delta x_i} \leq 1.$$

Downwind scheme is applicable for the right-hand boundary. It is also stable and it is not easy to incorporate.

Explicit forward difference discretization:

A method for computing the approximations to $u(x,t)$ at grid points in successive rows $u(x_i,t_j); i = 1,2,3,4, \dots\dots, n; j = 2,3, \dots\dots, m$ will be developed .The difference formula used for $u_t(x,t)$ and $u_{xx}(x,t)$ are

$$u_t(x,t) = \frac{u(x,t+k)-u(x,t)}{k} + 0(k)$$

And $u_{xx}(x,t) = \partial_x^+ \partial_x^- u(x,t)$

$$= \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}$$

The grid spacing is uniform in every row such that $x_{i+1} = x_i + h$ and its uniform in every column$t_{j+1} = t_j + k$.

Now, we drop the terms $0(k)$ and $0(h^2)$ and use the approximation $u_{i,j}$ for $u(x_i,t_j)$.Then the simplest numerical discretization of heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the explicit centered difference scheme

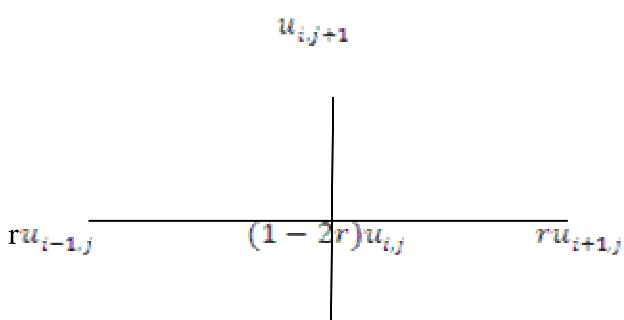
$$\frac{u_{i,j+1}-u_{i,j}}{k} = c^2 \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$$

$$\Rightarrow u_{i,j+1} = u_{i,j} + \frac{c^2 k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$\Rightarrow u_{i,j+1} = (1-2r)u_{i,j} + r(u_{i+1,j} + u_{i-1,j}); \text{ --------- (1)}$$

Where, $r=\frac{c^2 k}{h^2}$

This formula explicitly gives the value $u_{i,j+1}$ in terms of $u_{i+1,j}$, $u_{i,j}$ and $u_{i-1,j}$.The computational stencil is



If the error made at one stage of the calculation is eventually damped out, the method is called stable.

The explicit forward difference equation (1) is stable if and only if r is restricted to the interval $0 \leq r \leq \frac{1}{2}$.

Since the initial condition $u(x,0) = f(x)$ for each $0 \leq x \leq a$ implies that $u_{i,0} = f(x_i)$, for each $i = 0,1,2, \dots\dots, m$, these values can be used in equation (1) to find the value of $u_{i,1}$, for each $i = 1,2, \dots\dots, (m-1)$ The additional condition $u(0,t) = 0$ and $u(a,t) = 0$ imply that $u_{0,1} = u_{m,1}$ =0, so all the entries of the form $u_{i,1}$ can be determined. If the procedure is reapplied are all the approximations $u_{i,1}$ are known, the values of $u_{i,2}, u_{i,3}, \dots\dots$ can be obtained in a similar manner.

Numerical experiments and results:

We develop a computer program (code) in Matlab programming of scientific computing and implement the explicit forward difference discretization for a heat equation. We implement the explicit forward difference discretization, which has been presented in chapter 4 for the numerical simulation of a heat equation. The main parts of the implementation of our numerical scheme are given as in the following algorithm:

Input: nt and nx are the numbers of grid points of time and space respectively. k and h are the right end points of [0,k] and [0,h].
uo is as a initial condition and ua as a boundary condition.
Out put: u(t,x) is the solution matrix.

Initialization:

```
k=t(2)-t(1);
h=x(2)-x(1);
r=(c.^2*k)/(h.^2);
q=1-2*r
```

Step 1: Calculation of Analytic solution

```
r<=.5
for e=2:nt
    for f=2:nx
    z(e,f)=exp(-t(e))*sin(x(f));
  end
end
    end
```

Step 2: Calculation of numerical discretization

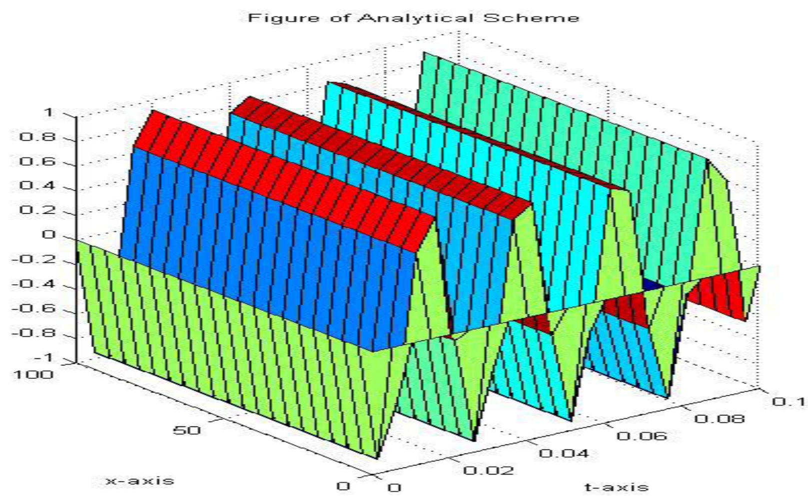
```
r<=.5
    for j=2:nt
        for i=2:nx-1
            u(j+1,i)=q*u(j,i)+r*(u(j,i+1)+u(j,i-1));
        end
    end
end
```

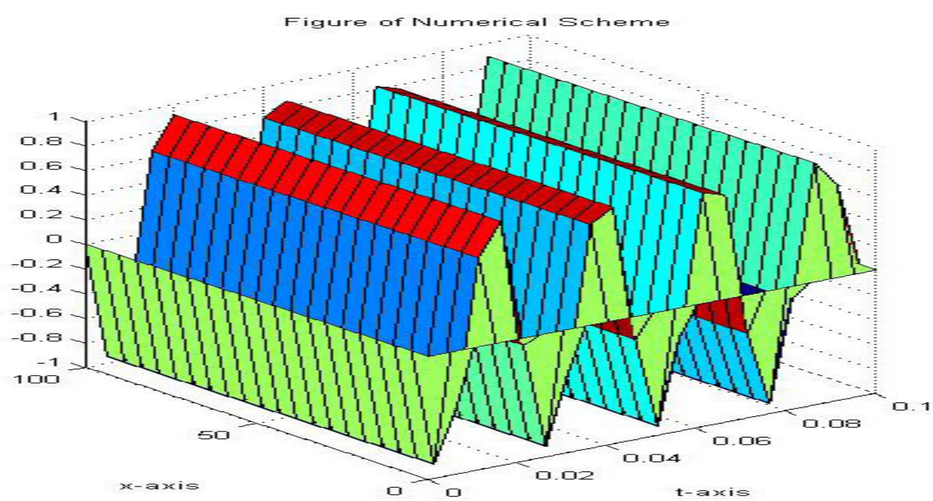
Step 3: Error estimation of Analytic solution and Numerical solution

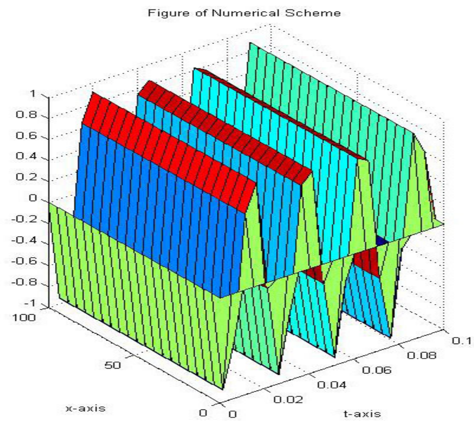
```
zab=abs(z);
zs=sum(zab');
zm=max(zs);
uab=abs(u);
us=sum(uab');
um=max(us);
w=z-u;
wab=abs(w);
ws=sum(wab');
wm=max(ws);
error=wm/zm*100
```

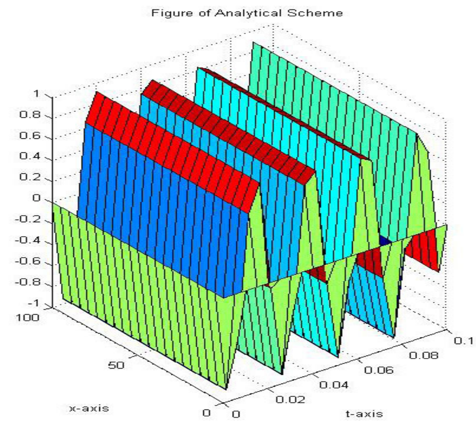
Step 4: Print u(t,x)
Step 5: Stop

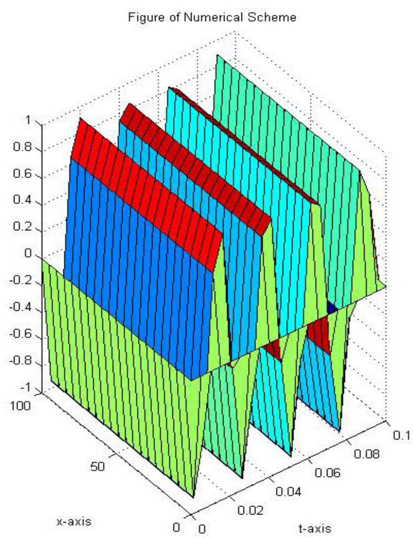
To test the accuracy of the implementation of the numerical scheme, we consider the heat equation. Now we show our results:

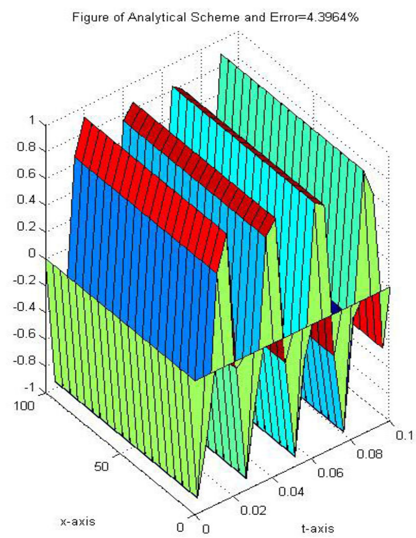












Conclusion:

In this project we have considered the second order heat equation. First we have shown that fundamentals of partial differential equation, heat equation, analytical solution by using separation of variable method. Finally we show the numerical result based on the explicit finite difference scheme agrees with basic some qualitative behavior of the model. From qualitative behavior we see that numerical solution is more applicable than analytic solution. This qualitative behavior agreement guarantees the implementation of heat equation model with sufficient accuracy. In future work, we implement the numerical scheme in higher order heat equation.

References:

[1] John A. Trangenstein, *"Numerical Solution of Partial Differential Equations",* February 22, 2000.

[2] L.S. Andallah "Analytical Method For First Order Cauchy Linear Partial Differential Equation and Finite Difference Method Explicit, Implicit Upwind, Explicit, Implicit Downwind Difference Scheme", *Research project report, Department of Mathematics*, Jahangirnagar University, 2008.

[3] Randall J. Le Veque, *"Numerical Methods For Conservation Laws",* Second Edition -1992, Birkhuser Verlag.

[4] M.D. Raisinghania, *"Ordinary and Partial Differential Equations"*.

[5] From Wikipedia, the free encyclopedia, "Differential Equations From Outside Physics".

[6] Richard Haberman, *"Mathematical Models",* Prentice-Hall, Inc.1977.

[7] Clive L.Dym, *"Principles of Mathematical modeling"* Academic press (ELSEVIER)-2004.

[8] Edward J.Beltrami, *"Mathematics for dynamic modeling"*.

[9] Duane Harselman, *Bruce Littlefield "Mastering Matlab"*.